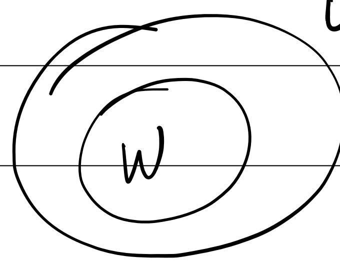
4.2 – Subspaces

Due Fri

Definition: A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem 4.2.1 Subspace Test
If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.
a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.
b) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.



V: obj, $+$, scalar mult, 10 axioms hold

Since any vector $\vec{w} \in W$ is also in V, vector space axioms 2, 3, 7, 8, 9, 10 hold. To show $W \subset V$ is itself a subspace, we need to confirm axioms 1, 4, 5, 6. Our work is simplified because if 6 holds, then $k = 0 \Rightarrow 0\vec{u} = \vec{0}$ so 4 holds. Also, if $k = -1$, $-1\vec{u} = -\vec{u}$ so 5 holds.

Example: (3) Use the Subspace Test to determine which of the sets are subspaces of M_{nn}.

a. The set of all diagonal $n \times n$ matrices.

b. The set of all $n \times n$ matrices A such that $\det(A) = 0$.

c. The set of all $n \times n$ matrices A such that $\mathrm{tr}(A) = 0$. sum of diagonal entries

d. The set of all symmetric $n \times n$ matrices. $A = A^T$

a. Let $\vec{u} = \begin{bmatrix} u_1 & & & 0 \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 & & & 0 \\ & v_2 & & \\ & & \ddots & \\ 0 & & & v_n \end{bmatrix}$, $k \in \mathbb{R}$

1. $\vec{u} + \vec{v} = \begin{bmatrix} u_1+v_1 & & & 0 \\ & u_2+v_2 & & \\ & & \ddots & \\ 0 & & & u_n+v_n \end{bmatrix}$, which is diagonal

2. $k\vec{u} = \begin{bmatrix} ku_1 & & & 0 \\ & ku_2 & & \\ & & \ddots & \\ 0 & & & ku_n \end{bmatrix}$, which is diagonal

This is a subspace.

$\det(A) + \det(B) \overset{?}{=} \det(A+B)$ No.

b. Let $A = \begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 & \\ & & & \ddots \\ 0 & & & & 0 \end{bmatrix}$

$\det(A) = 0$ $\quad \det(B) = 0$

$A + B = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$ $\quad \det(A+B) = 1 \neq 0$

This is not a subspace.

c. Let $U=[u_{ij}]$ & $V=[v_{ij}] \in M_{nn}$

$\ni tr(U)=0, tr(V)=0.$ Then $tr(U)=\sum_{i=1}^{n} u_{ii}$

and $tr(\vec{v})=\sum_{i=1}^{n} v_{ii}$. So $tr(U+V)=\sum_{i=1}^{n}(u_{ii}+v_{ii})$

$=\sum_{i=1}^{n} u_{ii}+\sum_{i=1}^{n} v_{ii}=0+0=0 \Rightarrow tr(U+V)=0$

Let $k \in \mathbb{R}$. $tr(kU)=\sum_{i=1}^{n} k u_{ii}=k\sum_{i=1}^{n} u_{ii}$

$=k(0)=0$

This is a subspace.

d. is on you when you make time.

Example: (6) Use the Subspace Test to determine which of the sets are subspaces of P_3.

a. All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.

b. All polynomials of the form $a_0 + a_1x$ in which a_0 and a_1 are real numbers.

a. No. If $k=\pi$, $\pi a_0+\pi a_1x+\pi a_2x^2+\pi a_3x^3$ but πa_i are not rational.

b. Let $\vec{p}=a_0+a_1x$ and $\vec{q}=b_0+b_1x$ and $k\in\mathbb{R}$.

$\vec{p}+\vec{q}=a_0+b_0+a_1x+b_1x=(a_0+b_0)+(a_1+b_1)x$, which has the form c_0+c_1x, $c_0,c_1\in\mathbb{R}$

and $k\vec{p}=ka_0+ka_1x$, which has the form d_0+d_1x, $d_0,d_1\in\mathbb{R}$.

This is a subspace.

Example: (11) Use the Subspace Test to determine which of the sets are subspaces of M_{22}.

a. All matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$.

b. All matrices of the form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$.

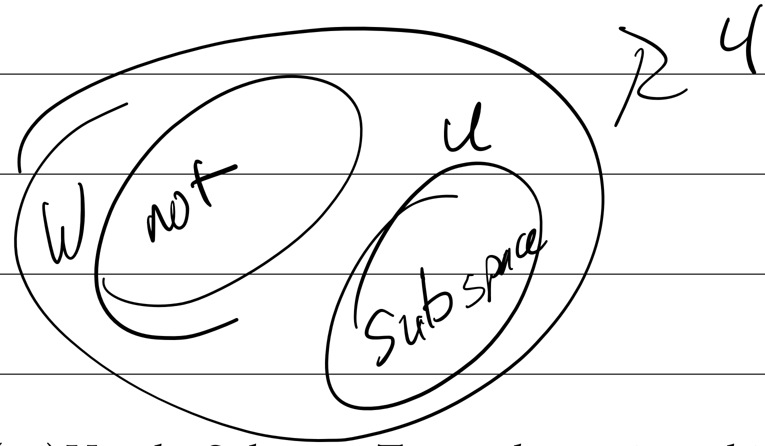
Let A & $B\in M_{22}$ ...

c. All 2×2 matrices A such that $\boxed{A\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$.

Let $A\in M_{22}$ be as described and $k\in\mathbb{R}$.

Then $k\boxed{A\begin{bmatrix} 1 \\ -1 \end{bmatrix}}=k\boxed{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}=\begin{bmatrix} 2k \\ 0 \end{bmatrix}\neq\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ if $k\neq 1$.

This is not a subspace.



Example: (14) Use the Subspace Test to determine which of the sets are subspaces of R^4.

W a. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $A = \begin{bmatrix} 0 & -1 & 0 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

U b. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where A is as in part (a).

a. Let $\vec{x}, \vec{y} \in R^4 \ni A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

This is not a subspace.

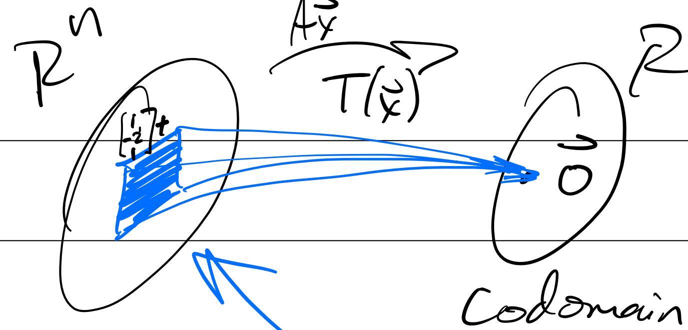
b. Let $\vec{x}, \vec{y} \in R^4 \ni A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\vec{y}$, $k \in R$.

Then $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A(k\vec{x}) = kA\vec{x} = k\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This is a subspace.

Generalizing: A similar process, applied to $A \in M_{mn}$, $\vec{x} \in R^n$, and $\vec{0} \in R^m$ will give a similar result. That is, $\{\vec{x} \in R^n \mid A\vec{x} = \vec{0}\}$ is a subspace.



in R^n in R^m

Theorem 4.2.3 The solution set of a homogeneous system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.

$f(x) = 2x \Rightarrow f = 2$

Definition: The solution set of a homogeneous system in n unknowns is a subspace of R^n, called the **solution space** of the system.

$T_A(\vec{x}) = A\vec{x}$ does not mean $T_A = A$ or $T_A = A\vec{x}$

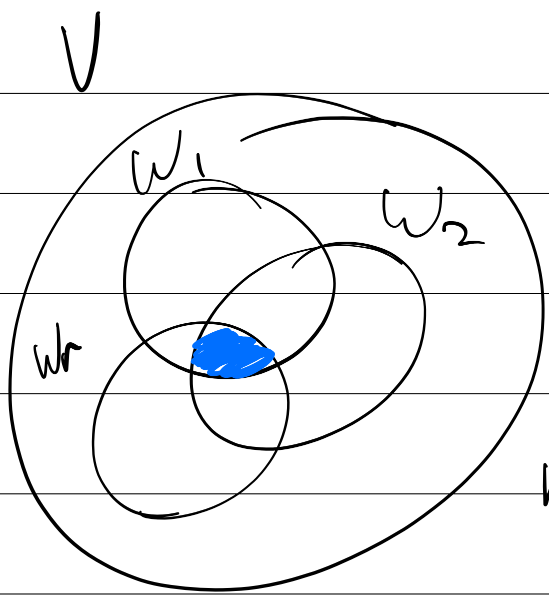
Definition: Let $T_A : R^n \to R^m$ be multiplication by the coefficient matrix A. The solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in R^n that T_A maps into the zero vector in R^m. This set is called the **kernel** of the transformation.

but If $T_A(\vec{x}) = A\vec{x}$, then $[T_A] = A$

Theorem 4.2.4 If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A : R^n \to R^m$ is a subspace of R^n.

Theorem 4.2.2 If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

nonempty, closed under addition & scalar mult.



$\vec{0} \in W_1, W_2, \ldots W_r$ because W_i are subspaces so $W_1 \cap W_2 \cap \cdots \cap W_r$ is not empty

Let $\vec{u}, \vec{v} \in W_1 \cap W_2 \cap \cdots \cap W_r$

Since each W_i is a subspace,

$\vec{u}+\vec{v} \in$ each $W_i \Rightarrow \vec{u}+\vec{v} \in \bigcap_{i=1}^{r} W_i$

Let $k \in \mathbb{R}$. Then $\vec{u} \in W_i \ \forall i$

$\Rightarrow k\vec{u} \in W_i \ \forall i. \Rightarrow k\vec{u} \in \bigcap_{i=1}^{r} W_i.$

So $W_1 \cap W_2 \cap \cdots \cap W_r$ is a subspace.

Examples of Subspaces

$\{\mathbf{0}\}$ is a subspace of every vector space V

Any vector space V is a subspace of itself

Subspace of R^2

Lines through the origin $\{(x,y) \mid ax+by=0\}$

Subspaces of R^3

Lines through the origin

Planes through the origin $\{(x,y,z) \mid ax+by+cz=0\}$

The solution space of a homogeneous system in n unknowns is a subspace of R^n

Subspaces of M_{nn}

Symmetric matrices

Triangular matrices

Diagonal matrices

Subspaces of $F(-\infty, \infty)$ (The following is actually a sequence of nested subspaces)

$C(-\infty, \infty)$, the set of functions continuous on R

$C^1(-\infty, \infty)$, the set of functions with continuous first-order derivatives on R

$C^n(-\infty, \infty)$, the set of functions with continuous n^{th}-order derivatives on R

$C^\infty(-\infty, \infty)$, the set of functions with derivatives of all orders on R

P_∞, the set of polynomials

P_n, polynomials of degree $\leq n$

$A\vec{x} = \vec{0}$: $[A \mid \vec{0}]$ Row reduce

to $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$ $\rightarrow x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$\rightarrow x_2 + 2x_3 = 0 \Rightarrow x_2 = 2x_3$

Let $x_3 = t$

$$\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$$

Solutions to

$A\vec{x} = \vec{0}$ look like, so

$$A \boxed{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t} = \vec{0}$$